Lecture 12

An Explicit Finite-Volume Algorithm with Multigrid

David W. Zingg

University of Toronto Institute for Aerospace Studies

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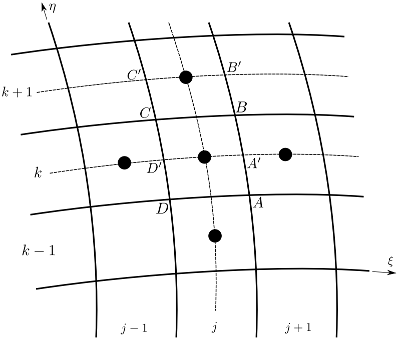
An Explicit Finite-Volume Algorithm with Multigrid

Key Characteristics

- cell-centered data storage; the numerical solution for the state variables is associated with the cells of the grid
- second-order finite-volume spatial discretization with added numerical dissipation; a simple shock-capturing device
- applicable to structured grids
- explicit multi-stage time marching with implicit residual smoothing and multigrid

Spatial Discretization: Cell-Centered Finite-Volume Method

Spatial Discretization: Cell-Centered Finite-Volume Method



Cell centered data storage

Spatial Discretization: Cell-Centered Finite-Volume Method

Integral form of conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} Q\mathrm{d}V + \oint_{S(t)} \hat{n}\cdot\mathcal{F}\mathrm{d}S = \int_{V(t)} P\mathrm{d}V$$

2D, no source terms:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{A} Q\mathrm{d}A + \oint_{C} \hat{n}\cdot\mathcal{F}\mathrm{d}l = 0$$

Cartesian coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{A} Q\mathrm{d}A + \oint_{C} \hat{n}\cdot(E\hat{i}+F\hat{j})\mathrm{d}l = \oint_{C} \hat{n}\cdot(E_{\mathrm{v}}\hat{i}+F_{\mathrm{v}}\hat{j})\mathrm{d}l$$

$$\hat{n}\mathrm{d}l = \mathrm{d}y\hat{i} - \mathrm{d}x\hat{j}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{A} Q\mathrm{d}A + \oint_{C} (E\mathrm{d}y - F\mathrm{d}x) = \oint_{C} (E_{\mathrm{v}}\mathrm{d}y - F_{\mathrm{v}}\mathrm{d}x)$$

Spatial Discretization: Cell-Centered Finite-Volume Method

Semi-discrete form:

$$A_{j,k}\frac{\mathrm{d}}{\mathrm{d}t}Q_{j,k} + \mathcal{L}_{\mathrm{i}}Q_{j,k} + \mathcal{L}_{\mathrm{ad}}Q_{j,k} = \mathcal{L}_{\mathrm{v}}Q_{j,k}$$

Cell average:

$$Q_{j,k} = \frac{1}{A_{j,k}}\int_{A_{j,k}} Q\mathrm{d}A$$

Inviscid and Viscous Fluxes

Inviscid operator (second order):

$$\mathcal{L}_{\mathrm{i}} Q = \sum_{l=1}^{4} (\mathcal{F}_{\mathrm{i}})_l \cdot \mathbf{s}_l$$

$$\mathbf{s}_l = (\Delta y)_l \hat{i} - (\Delta x)_l \hat{j}$$

$$(\mathcal{F}_{\mathrm{i}})_l = \frac{1}{2}(\mathcal{F}_{\mathrm{i}}^{-} + \mathcal{F}_{\mathrm{i}}^{+}) = \frac{1}{2}(Q^{-}\mathbf{v}^{-} + Q^{+}\mathbf{v}^{+})_l + \bar{\mathcal{P}}_l$$

$$\bar{\mathcal{P}}_l = [\ 0, \quad \frac{1}{2}(p^{-} + p^{+})_l \hat{i}, \quad \frac{1}{2}(p^{-} + p^{+})_l \hat{j}, \quad \frac{1}{2}(p^{-}\mathbf{v}^{-} + p^{+}\mathbf{v}^{+})_l\ ]^T$$

Inviscid and Viscous Fluxes

Viscous operator (second order)

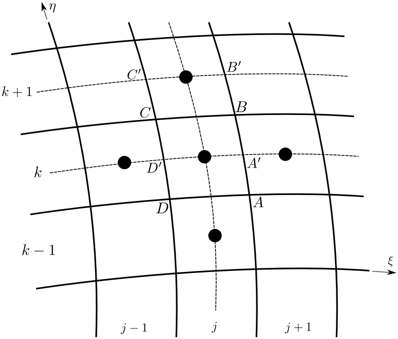
Viscous flux tensor contains velocity derivatives

$$\int_A \nabla Q \mathrm{d}A = \oint_C \hat{n} Q \mathrm{d}l$$

$$\begin{aligned} \int_{A'} \frac{\partial u}{\partial x} \mathrm{d}A &= \oint_{C'} u \mathrm{d}y \\ \int_{A'} \frac{\partial u}{\partial y} \mathrm{d}A &= -\oint_{C'} u \mathrm{d}x \end{aligned}$$

$$\mathcal{L}_{\mathrm{v}} Q = \sum_{l=1}^{4} (\mathcal{F}_{\mathrm{v}})_l \cdot \mathbf{s}_l$$

Inviscid and Viscous Fluxes



Auxiliary cell A′B′C′D′ for computing viscous fluxes

Iteration to Steady State

Mechanism for converging to steady state

- The "error" is removed by 1) convecting out through the far-field boundary and 2) through dissipation

Multi-Stage Time-Marching Method

Multi-Stage Time-Marching Method

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_{j,k} = -\frac{1}{A_{j,k}}\mathcal{L}Q_{j,k}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_{j,k} = -\frac{1}{A_{j,k}}(\mathcal{L}_{\mathrm{i}} + \mathcal{L}_{\mathrm{ad}})Q_{j,k} = -R(Q_{j,k})$$

Multi-stage method with q stages:

$$\begin{aligned} Q_{j,k}^{(0)} &= Q_{j,k}^{(n)} \\ Q_{j,k}^{(m)} &= Q_{j,k}^{(0)} - \alpha_m h R(Q_{j,k}^{(m-1)}) \qquad m = 1, \ldots, q \\ Q_{j,k}^{(n+1)} &= Q_{j,k}^{(q)} \end{aligned}$$

Multi-Stage Time-Marching Method

$\lambda - \sigma$ relation for 5-stage method:

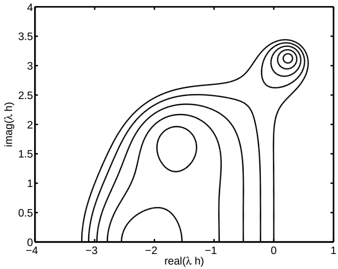
$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u$$

$$u_n = u_0 \sigma^n$$

$$\sigma = 1 + \beta_1 \lambda h + \beta_2 (\lambda h)^2 + \beta_3 (\lambda h)^3 + \beta_4 (\lambda h)^4 + \beta_5 (\lambda h)^5$$

$$\begin{aligned}
\beta_1 &= \alpha_5 \\
\beta_2 &= \alpha_5 \alpha_4 \\
\beta_3 &= \alpha_5 \alpha_4 \alpha_3 \\
\beta_4 &= \alpha_5 \alpha_4 \alpha_3 \alpha_2 \\
\beta_5 &= \alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1
\end{aligned}$$

Multi-Stage Time-Marching Method



Contours of $|\sigma|$ for the five-stage time-marching method with $\beta_3 = 1/6$, $\beta_4 = 1/24$, and $\beta_5 = 1/120$. Contours shown have $|\sigma|$ equal to 1, 0.8, 0.6, 0.4, and 0.2.

Multi-Stage Time-Marching Method

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

2nd-order centered differences with 3rd-order artificial dissipation ($a \geq 0$)

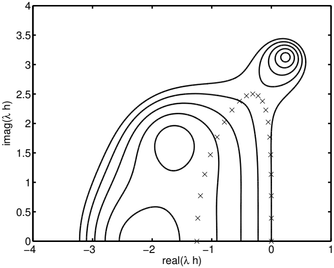
$$-a\delta_x u = -\frac{a}{\Delta x}\left[\frac{u_{j+1} - u_{j-1}}{2} + \kappa_4(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})\right]$$

Eigenvalues:

$$\lambda_m = -\frac{a}{\Delta x}\left\{i\sin\left(\frac{2\pi m}{M}\right) + 4\kappa_4\left[1 - \cos\left(\frac{2\pi m}{M}\right)\right]^2\right\} \quad m = 0 \ldots M-1$$

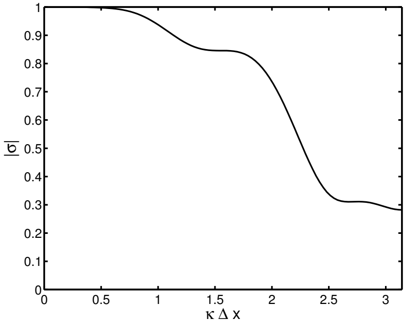
Multi-Stage Time-Marching Method

$$\lambda_m h = -C_\mathrm{n}\left\{ i\sin\left(\frac{2\pi m}{M}\right) + 4\kappa_4\left[1-\cos\left(\frac{2\pi m}{M}\right)\right]^2\right\} \quad m = 0\ldots M-1$$



Plot of λh values given for $M = 40$, $\kappa_4 = 1/32$, and $C_\mathrm{n} = 2.5$ with contours of $|\sigma|$ for the five-stage time-marching method with $\alpha_1 = 1/5$, $\alpha_2 = 1/4$, and $\alpha_3 = 1/3$. Contours shown have $|\sigma|$ equal to 1, 0.8, 0.6, 0.4, and 0.2.

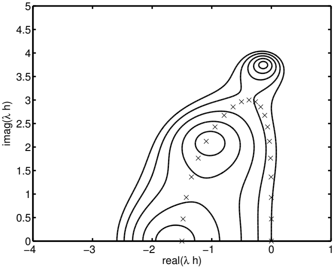
Multi-Stage Time-Marching Method



Plot of $|\sigma|$ values vs. $\kappa\Delta x$ for the spatial operator with $C_{\mathrm{n}} = 2.5$, $\kappa_4 = 1/32$, and the five-stage time-marching method with $\alpha_1 = 1/5$, $\alpha_2 = 1/4$, and $\alpha_3 = 1/3$.

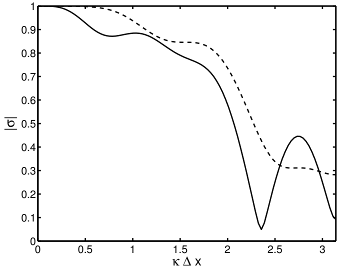
Multi-Stage Time-Marching Method

Change the α_m values (and the Courant number):



Plot of λh values for $M = 40$, $\kappa_4 = 1/32$, and $C_\mathrm{n} = 3$ with contours of $|\sigma|$ for the five-stage time-marching method with $\alpha_1 = 1/4$, $\alpha_2 = 1/6$, and $\alpha_3 = 3/8$.

Multi-Stage Time-Marching Method



Plot of $|\sigma|$ values vs. $\kappa\Delta x$ for the spatial operator with $C_{\rm n} = 3$, $\kappa_4 = 1/32$, and the five-stage time-marching method with $\alpha_1 = 1/4$, $\alpha_2 = 1/6$, and $\alpha_3 = 3/8$ (solid line). The dashed line shows the results with $C_{\rm n} = 2.5$ and $\alpha_1 = 1/5$, $\alpha_2 = 1/4$, and $\alpha_3 = 1/3$.

Multi-Stage Time-Marching Method

Consider computing the artificial dissipation only on stages 1, 3, and 5:

$$R^{(m-1)} = \frac{1}{A}\left(\mathcal{L}_{\mathrm{i}} Q^{(m-1)} + \sum_{p=0}^{m-1} \gamma_{mp} \mathcal{L}_{\mathrm{ad}} Q^{(p)}\right)$$

$$\gamma_{10} = 1$$

$$\gamma_{20} = 1, \quad \gamma_{21} = 0$$

$$\gamma_{30} = 1 - \Gamma_3, \quad \gamma_{31} = 0, \quad \gamma_{32} = \Gamma_3$$

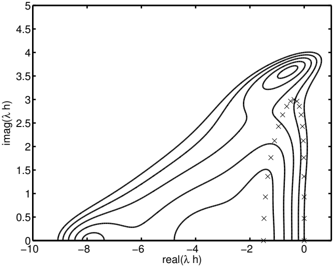
$$\gamma_{40} = 1 - \Gamma_3, \quad \gamma_{41} = 0, \quad \gamma_{42} = \Gamma_3, \quad \gamma_{43} = 0$$

$$\gamma_{50} = (1 - \Gamma_3)(1 - \Gamma_5), \quad \gamma_{51} = 0, \quad \gamma_{52} = \Gamma_3(1 - \Gamma_5), \quad \gamma_{53} = 0, \quad \gamma_{54} = \Gamma_5$$

Compute viscous terms on stage 1 only:

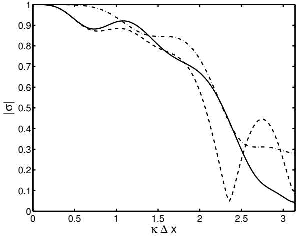
$$R^{(m-1)} = \frac{1}{A}\left(\mathcal{L}_{\mathrm{i}} Q^{(m-1)} - \mathcal{L}_{\mathrm{v}} Q^{(0)} + \sum_{p=0}^{m-1} \gamma_{mp} \mathcal{L}_{\mathrm{ad}} Q^{(p)}\right)$$

Multi-Stage Time-Marching Method



Plot of λh values for $M = 40$, $\kappa_4 = 1/32$, and $C_\mathrm{n} = 3$ with contours of $|\sigma|$ for the five-stage time-marching method with $\alpha_1 = 1/4$, $\alpha_2 = 1/6$, and $\alpha_3 = 3/8$ with the artificial dissipation computed only on stages 1, 3, and 5.

Multi-Stage Time-Marching Method



Plot of $|\sigma|$ values vs. $\kappa\Delta x$ for the spatial operator with $C_{\rm n} = 3$, $\kappa_4 = 1/32$, and the five-stage time-marching method with $\alpha_1 = 1/4$, $\alpha_2 = 1/6$, and $\alpha_3 = 3/8$ with the artificial dissipation computed only on stages 1, 3, and 5 (solid line). The dashed line shows the results with the artificial dissipation computed at every stage, and the dash-dot line shows the results with $C_{\rm n} = 2.5$ and $\alpha_1 = 1/5$, $\alpha_2 = 1/4$, and $\alpha_3 = 1/3$.

Multi-Stage Time-Marching Method

Local Time Stepping

For example, for inviscid flow:

$$(\Delta t)_j = \frac{(\Delta x)_j}{(|u| + a)_j} C_{\mathrm{n}}$$

Local time stepping is essential for fast convergence of an explicit scheme

For an explicit scheme, must be done more carefully than for an implicit scheme

Does not address the problem of cells with high aspect ratios

Multi-Stage Time-Marching Method

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

$$\lambda_m = -\frac{a}{\Delta x} i \sin\left(\frac{2\pi m}{M}\right) - \frac{4\nu}{\Delta x^2}\sin^2\left(\frac{\pi m}{M}\right) \quad m = 0, \ldots, M-1$$

$$\lambda_m h = -C_\mathrm{n} i \sin\left(\frac{2\pi m}{M}\right) - 4V_\mathrm{n}\sin^2\left(\frac{\pi m}{M}\right) \quad m = 0, \ldots, M-1$$

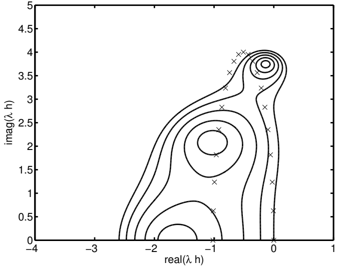
$$\begin{aligned} C_\mathrm{n} &= \frac{ah}{\Delta x} \leq 4 \\ V_\mathrm{n} &= \frac{\nu h}{\Delta x^2} \leq \frac{2.59}{4} \end{aligned}$$

Multi-Stage Time-Marching Method

$$h_{\mathrm{c}} \leq \frac{4\Delta x}{a}$$

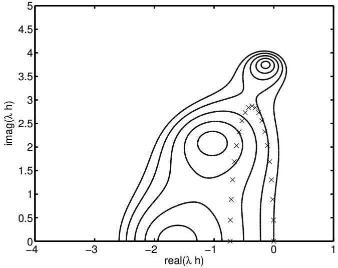
$$h_{\mathrm{d}} \leq \frac{2.59\Delta x^2}{4\nu}$$

Multi-Stage Time-Marching Method



Plot of λh values for $M = 40$, $\kappa_4 = 1/32$ with contours of $|\sigma|$ for the five-stage time-marching method with $\alpha_1 = 1/4$, $\alpha_2 = 1/6$, and $\alpha_3 = 3/8$. Time step based on minimum of h_c and h_d.

Multi-Stage Time-Marching Method

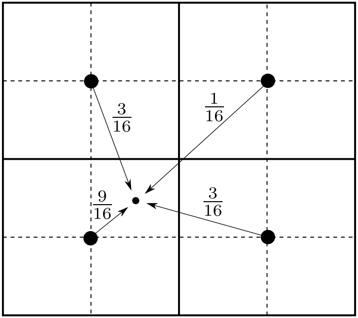


Plot of λh values for $M = 40$, $\kappa_4 = 1/32$ with contours of $|\sigma|$ for the five-stage time-marching method with $\alpha_1 = 1/4$, $\alpha_2 = 1/6$, and $\alpha_3 = 3/8$.

Time step based on
$$\frac{1}{h} = \frac{1}{h_{\mathrm{c}}} + \frac{1}{h_{\mathrm{d}}}$$

The Multigrid Method

The Multigrid Method



Coarse mesh construction for a cell-centered scheme

Recursive approach – so we will discuss a two-grid problem first

The Multigrid Method

ODE system on fine mesh

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{j,k} = -\frac{1}{A_{j,k}} \mathcal{L} Q_{j,k} = -R_{j,k}$$

Restrict residual to coarse mesh

$$I_h^{2h} R_h = \frac{1}{A_{2h}} \sum_{p=1}^{4} A_h R_h$$

Restrict solution to coarse mesh

$$Q_{2h}^{(0)} = I_h^{2h} Q_h = \frac{1}{A_{2h}} \sum_{p=1}^{4} A_h Q_h$$

The Multigrid Method

ODE on coarse mesh

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{2h} = -[R_{2h}(Q_{2h}) + P_{2h}]$$

Source term

$$P_{2h} = I_h^{2h} R_h - R_{2h}(Q_{2h}^{(0)})$$

Drives the following to zero:

$$R_{2h}(Q_{2h}) + P_{2h} = R_{2h}(Q_{2h}) - R_{2h}(Q_{2h}^{(0)}) + I_h^{2h} R_h$$

The Multigrid Method

At first stage

$$-[R_{2h}(Q_{2h}^{(0)}) + P_{2h}] = -[R_{2h}(Q_{2h}^{(0)}) + I_h^{2h} R_h - R_{2h}(Q_{2h}^{(0)})] = -I_h^{2h} R_h$$

At mth stage

$$Q_{2h}^{(m)} \quad = \quad Q_{2h}^{(0)} - \alpha_m h[R(Q_{2h}^{(m-1)}) + P_{2h}]$$

When restricting to the next coarser mesh level, the source term must be included

Condition on restriction and prolongation operators:

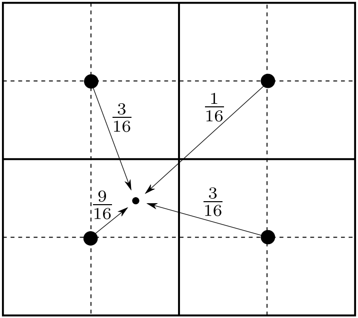
$$p_{\mathrm{R}} + p_{\mathrm{P}} + 2 > p_{\mathrm{PDE}}$$

For example, $p_{\mathrm{R}} = 0$, $p_{\mathrm{P}} = 1$ for Navier-Stokes equations ($p_{\mathrm{PDE}} = 2$)

Prolongation operator with $p_{\mathrm{P}} = 1$ (see figure):

$$I_{2h}^{h} \Delta Q = \frac{1}{16}(9\Delta Q_1 + 3\Delta Q_2 + 3\Delta Q_3 + \Delta Q_4)$$

The Multigrid Method



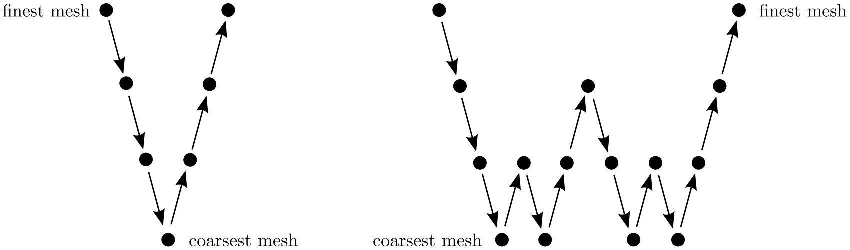
Bilinear prolongation operator for cell-centered scheme in two dimensions

The Multigrid Method

Prolong correction from coarse mesh to fine mesh:

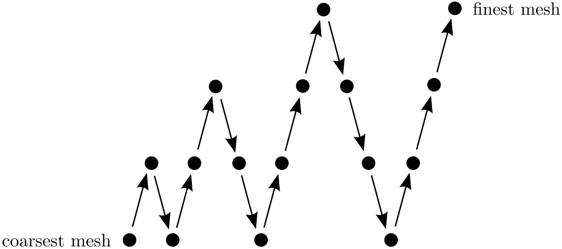
$$Q_h^{(\text{corrected})} = Q_h + I_{2h}^h (Q_{2h} - Q_{2h}^{(0)})$$

The Multigrid Method



Four-grid V and W multigrid cycles

The Multigrid Method



Full multigrid with four grids